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Classical charged particles with spin

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Abstract. A Lagrangian formulation for a relativistic classical spinning point particle interacting with the electromagnetic field is given. Although it meets all criteria usually imposed for this system, it is nevertheless unrealistic, in that the centres of mass and charge get separated in electromagnetic fields. It is suggested that unless special care is taken, the same can happen also in different treatments of the problem.

With the development of supersymmetry, the venerable subject of classical spinning particles (Corben 1968, Barut 1964, Itzykson and Voros 1972, Souriau 1970) has also got a new impetus (Berezin and Marinov 1976, Collins and Tucker 1976, Barducci *et al* 1976, Brink *et al* 1977). Although the problem of classical relativistic spinning point particles in an electromagnetic field has at least 50 years of history (Frenkel 1926, Thomas 1927), only the case of weak and homogeneous fields has been solved (Bargman *et al* 1959, Barut 1964), but a fully satisfactory Lagrangian formalism does not exist.

It is the purpose of this paper to present a Lagrangian not using supersymmetry, which yields in the limit of weak and homogeneous fields the Bargmann–Michel–Telegdi equations (Bargman *et al* 1959). In general fields, however, it describes a rather curious particle, in which the centre of gravity and the centre of charge do not coincide, although the particle otherwise is pointlike. This happens not only when the particle traverses the field, but remains so even after it has passed through a strong field.

It seems that this default is indeed common also to previous attempts, although none of these is sufficiently far developed to make it evident.

Our action is (we use $c = 1$)

$$W = \int d^4x \left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \int_{-\infty}^{+\infty} d\tau \, . \, L \, . \, \delta^4(x - z(\tau)) \right) \tag{1}$$

with

$$L = \tfrac{1}{2}m(1 - u_\mu u^\mu) - eu_\mu A^\mu(z) + \tfrac{1}{2}(\dot{b}_\mu a^\mu - \dot{a}_\mu b^\mu) + \kappa a_\mu b_\nu F^{\nu\mu}(z) + u_\mu(\beta b^\mu - \alpha a^\mu). \tag{2}$$

Here, $u_\mu \equiv dz_\mu/d\tau \equiv \dot{z}_\mu$, and all quantities, except e and κ but including the mass m, are functions of τ. The first term in equation (2) is indeed a constraint added (with m as Lagrangian multiplier) to yield

$$2\frac{\partial L}{\partial m} = 1 - u^2 = 0. \tag{3}$$

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Similarly, α and β are multipliers whose variation guarantees that a_μ and b_μ are orthogonal on u_μ,

$$au = bu = 0. \tag{4}$$

Variation with respect to a_μ and b_μ gives finally

$$\dot{a}_\mu = \kappa F_{\mu\nu} a^\nu + \beta u_\mu \tag{5a}$$

$$\dot{b}_\mu = \kappa F_{\mu\nu} b^\nu + \alpha u_\mu. \tag{5b}$$

Spin is described by the antisymmetric tensor

$$S_{\mu\nu} = a_\mu b_\nu - a_\nu b_\mu \tag{6}$$

which by equation (4) is also orthogonal on u,

$$S_{\mu\nu} u^\nu = 0. \tag{7}$$

Its proper time derivative is

$$\dot{S}_{\mu\nu} = \kappa (F_{\mu\alpha} S^\alpha{}_\nu - F_{\nu\alpha} S^\alpha{}_\mu) - u_\mu \eta_\nu + u_\nu \eta_\mu, \tag{8}$$

with

$$\eta_\mu = \alpha a_\mu - \beta b_\mu. \tag{9}$$

From this we see that

$$S_{\mu\nu} S^{\mu\nu} = 2\sigma^2 = \text{constant}, \tag{10}$$

so that the Lagrangian indeed describes a particle with a constant magnitude of spin.

The equation of motion for the four-velocity is finally

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(mu_\mu + \eta_\mu) = eF_{\mu\nu} u^\nu + \tfrac{1}{2}\kappa S_{\alpha\beta} F^{\alpha\beta}{}_{,\mu}. \tag{11}$$

Except maybe for the term $\dot{\eta}_\mu$, this is just what we expect for a particle with $g = 2m\kappa/e$. Multiplying it with u_μ, we get

$$m(\tau) = m_0 + \tfrac{1}{2}\kappa (S_{\alpha\beta} F^{\alpha\beta}), \tag{12}$$

where the constant m_0 is the mass in field-free regions, and where we have used

$$u_\mu \eta^\mu = 0 \tag{13}$$

together with equations (4) and (5). The fact that the effective particle mass acquires a term proportional to (S . F) is indeed well known (Corben 1968, Barut 1964).†

A further constraint on η is obtained by taking the derivative of equation (7):

$$m\eta_\mu - S_{\mu\nu}\dot{\eta}^\nu = (m\kappa - e) S_{\mu\nu} F^{\nu\alpha} u_\alpha - \tfrac{1}{2}\kappa S_{\mu\nu} S_{\alpha\beta} F^{\alpha\beta,\nu}. \tag{14}$$

For a weak homogeneous field $F_{\mu\nu}$ and for $\kappa = e/m_0$ (corresponding to $g = 2$), this has the special solution $\eta = 0$, yielding the well known result that four-velocity and spin rotate with the same 'angular velocity' $e/mF_{\mu\nu}$. For the following discussion it is useful to introduce the Pauli-Lubanski spin vector

$$S_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma} u^\nu s^{\rho\sigma}, \qquad \text{with } S^2 = -\sigma^2. \tag{15}$$

†Notice that in Barut (1964) the roles of m and m_0 are interchanged. In an alternative formulation, one might demand $\kappa(\tau) = eg/2m(\tau)$. Then, one would get $m(\tau) = m_0 = \text{constant}$, but $u^2 = 1 + (eg/2m^2) S_{\mu\nu} F^{\mu\nu}$. The problems to be discussed later would persist.

Its τ-derivative is found to be

$$\dot{S}_\mu = \kappa F_{\mu\nu} S^\nu - \left(\frac{e}{m} - \kappa\right) u_\mu S_\nu F^{\nu\lambda} u_\lambda - \frac{\kappa}{2m} u_\mu S_{\alpha\beta} F^{\alpha\beta,\nu} S_\nu + \frac{1}{m} u_\mu S_\nu \dot{\eta}^\nu, \tag{16}$$

the first terms of which are just the BMT-equation (Bargman *et al* 1959). It is orthogonal both on u and on η,

$$S_\mu u^\mu = 0, \tag{17}$$

$$S_\mu \eta^\mu = 0, \tag{18}$$

the latter following from equations (6) and (9).

Summarising we see that we can forget about the auxiliary fields a_μ, b_μ, α and β. The particle is completely described by its position z and the *vierbein*† $u_\mu = \dot{z}_\mu$, $\hat{S}_\mu \equiv S_\mu/\sqrt{-S^2}$, $\hat{\eta}_\mu \equiv \eta_\mu/\sqrt{-\eta^2}$ and $t_\mu = \epsilon_{\mu\nu\rho\sigma} u^\nu \hat{S}^\rho \hat{\eta}^\sigma = S_{\mu\sigma}\hat{\eta}^\sigma/\sqrt{-S^2}$. While the evolution equations of z, u, and S are explicit, equation (14) for η (and thus also the equation for t) is only implicit. Fortunately, however, the projections of η on each of the vectors of the *vierbein* can be computed from equations (14) and (11), yielding the rather lengthy result

$$\begin{aligned}\dot{\eta}_\mu = {} & \kappa u_\mu (u_\alpha F^{\alpha\beta} \eta_\beta) - \kappa \hat{S}_\mu (\hat{S}_\alpha F^{\alpha\beta} \eta_\beta) - \hat{\eta}_\mu [(e - \kappa m)\hat{\eta}_\alpha F^{\alpha\nu} u_\nu + \tfrac{1}{2}\kappa S_{\alpha\beta} F^{\alpha\beta,\nu} \hat{\eta}_\nu] \\ & - t_\mu [(e - \kappa m) t_\alpha F^{\alpha\nu} u_\nu + \tfrac{1}{2}\kappa S_{\alpha\beta} F^{\alpha\beta,\nu} t_\nu] + m S_{\mu\nu} \eta^\nu / s^2. \end{aligned} \tag{19}$$

Up to now, our results perhaps looked somewhat complicated (but this was to be expected), but quite reasonable. The blow comes when we consider the passage of our particle through a space–time region where $F_{\mu\nu} \neq 0$, with $F_{\mu\nu} = 0$ outside this region. Even if $\eta_\mu \to 0$ for $\tau \to -\infty$, equation (19) leads in general to $\eta_\mu \neq 0$ for the outgoing particle.

This is most easily seen by looking at the norm of η, whose derivative is not a total derivative of a function of z, u, S, and F:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \eta^2 = 2(e - m\kappa)\eta_\alpha F^{\alpha\nu} u_\nu + \kappa S_{\alpha\beta} F^{\alpha\beta,\nu} \eta_\nu. \tag{20}$$

After leaving the field, η keeps its length and rotates, according to

$$\dot{\eta}_\mu = -\frac{m}{\sigma^2} S_{\mu\nu} \eta^\nu \tag{21}$$

with an angular velocity $m/\sqrt{\sigma^2}$, around the spin axis. As a result, $u_\mu \neq$ constant even after the particle has left the field behind it! (This might be changed by radiation damping, but it also seems unacceptable that a point particle radiates *after* it has passed an external field.) In contrast, what is constant for $F_{\mu\nu} = 0$ is the canonical momentum

$$p_\mu = \frac{\partial L}{\partial u^\mu} = -(m u_\mu + \eta_\mu + e A_\mu), \tag{22}$$

†We might mention that $S_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} S^\rho u^\sigma$.

and the position $z_\mu(\tau)$ spirals along the straight line

$$\begin{aligned} X_\mu(\tau) &= \int^\tau \mathrm{d}\tau' \Big(u_\mu(\tau') + \frac{1}{m(\tau')}\eta_\mu(\tau')\Big) \\ &\underset{(\text{For } F=0)}{=} \frac{\tau}{m} p_\mu + \text{constant}. \end{aligned} \tag{23}$$

It is tempting to interpret $X_\mu(\tau)$ as the centre-of-mass position of the particle, while $z_\mu(\tau)$ by construction is the position of the (point-like) charge.

(The fact that the charge distribution is pointlike is most easily seen from the current

$$j_\mu(x) = \partial^\nu F_{\nu\mu}(x) = \int_{-\infty}^{+\infty} \mathrm{d}\tau (e u_\mu(\tau) + \kappa S_{\mu\nu}\partial^\nu)\delta^4(x - z(\tau)). \qquad (24))$$

This is supported by the observation that $S_{\mu\nu}$ is not constant either. The constant (for $F=0$) spin tensor is

$$\Sigma_{\mu\nu} = S_{\mu\nu} - [(X_\mu - z_\mu)p_\nu - (X_\nu - z_\nu)p_\mu], \tag{25}$$

where the term in the square bracket is obviously the orbital angular momentum around the point z.

The difference between centre-of-mass z and centre-of-charge X could arise since we had imposed no constraint forcing them to be identical, while we had at the same time auxiliary variables a_μ, b_μ, α and β which are not completely eliminated. (We should mention that the non-relativistic limit is free of these diseases. In this limit, we take account of the constraints $a_\mu u^\mu = b_\mu u^\mu = 0$ by considering a_0 and b_0 as dependent variables $a_0 \approx \boldsymbol{a}\,.\,\boldsymbol{v}/c$, $b_0 \approx \boldsymbol{b}\,.\,\boldsymbol{v}/c$. After restoring factors of c, we arrive at the Lagrangian, correct including order v/c,

$$L^{\mathrm{nr}} = \frac{m}{2}\boldsymbol{v}^2 - e\phi + \frac{e}{c}\boldsymbol{v}\,.\,\boldsymbol{A} + \tfrac{1}{2}(\boldsymbol{b}\,.\,\dot{\boldsymbol{a}} - \dot{\boldsymbol{b}}\,.\,\boldsymbol{a}) + (\boldsymbol{a}\times\boldsymbol{b})\left[\frac{ge}{2mc}\Big(\boldsymbol{B} - \frac{\boldsymbol{v}}{c}\times\boldsymbol{E}\Big) + \frac{\boldsymbol{v}\times\dot{\boldsymbol{v}}}{2c^2}\right].$$

The last contribution, resulting from the term $\frac{1}{2}(\dot{b}_0 a_0 - \dot{a}_0 b_0)$ in equation (2), is the famous Thomas term. It can be neglected in the equation of motion for $\boldsymbol{v}$, but not in the equation for the spin $\boldsymbol{S} = \boldsymbol{a}\times\boldsymbol{b}$.)

Our crucial observation is that none of the previous treatments (Corben 1968, Barut 1964, Itzykson and Voros 1972, Souriau 1970) imposed this constraint either, though they also used auxiliary variables†. So we cannot exclude a similar paradox in these theories as well. In that of Barut (1964) it definitely does occur, since our model is just a specific realisation of the general class of models discussed there. For the other treatments, the situation is less clear. In Itzykson and Voros (1972), e.g., no attempt was made to check the consistency of the constraints (3) and (7), and in our model it was just these constraints which forced us to introduce auxiliary variables in the form of Lagrangian multipliers. Anyhow, none of the above models has been developed far enough that the above paradox could have been seen.

That the centres of mass and charge do not necessarily coincide for spinning particles is indeed well known (Corben 1968). However, the fact that $X_\mu = z_\mu$ is a particular solution in field-free regions seems to have misled many authors to believe that $X_\mu \neq z_\mu$ appears only in external fields. There, it would not be disturbing. What

† That this excess of numbers of variables over the numbers of degrees of freedom is the source of troubles with classical (and quantised!) spinning particles was stressed by Souriau (1966) and Bacry (1967, 1976).

we think is disturbing and unnoticed before, is the observation that $X_\mu = z_\mu$ is in general impossible before *and* after passage through a field, at least in our particular model.

The most straightforward way to guarantee that the centres of mass and charge coincide would consist in treating particles coupled to electromagnetic *and* gravitational fields. We expect that weak gravitation fields are coupled to an energy–momentum tensor

$$T_{\mu\nu}(x) = \int_{-\infty}^{+\infty} \mathrm{d}\tau \, [m u_\mu u_\nu + \tfrac{1}{2}(u_\mu S_{\nu\alpha}\partial^\alpha + u_\nu S_{\mu\alpha}\partial^\alpha)] \, \delta^4(x - z(\tau)) + T^{\mathrm{el}}_{\mu\nu}(x), \tag{26}$$

(where el stands for electromagnetic) which should be conserved when neglecting gravitation. This is not the case in our model. There, the generators of the Poincaré group are

$$P_\mu = m u_\mu + \eta_\mu + P^{\mathrm{el}}_\mu$$

and

$$M_{\mu\nu} = z_\mu P_\nu - z_\nu P_\mu + S_{\mu\nu} + M^{\mathrm{el}}_{\mu\nu} \tag{27}$$

and one can verify explicitly that they are constants of the motion for the action given by equations (1) and (2). Attempts along this line have not yet been successful.

Finally, let us make some comments about theories describing spin by Grassmann variables (Berezin and Marinov 1976, Collins and Tucker 1976, Barducci *et al* 1976, Brink *et al* 1977). There, one introduces a real Grassmann vector ξ_μ which commutes with the coordinate z_μ, but anticommutes with itself,

$$\xi_\mu \xi_\nu = -\xi_\nu \xi_\mu. \tag{28}$$

The spin tensor is

$$S_{\mu\nu} = \propto \mathrm{i} \xi_\mu \xi_\nu \tag{29}$$

which by equation (28) is antisymmetric but which, also by equation (28), satisfies

$$(S_{\mu\nu})^2 = 0 \qquad \text{(no summation)}, \tag{30}$$

and $S_\mu S^\mu = -\sigma^2 = 0$. The physical interpretation of such theories is somewhat delicate. Presumably, one should interpret them as the limit $\hbar \to 0$ of theories where $\boldsymbol{S} \propto \hbar$.

If we neglect in the present formalism all terms quadratic in $S_{\mu\nu}$ we can choose $\eta_\mu = O(S_{\alpha\beta})$, and we also get a consistent and correct result, provided $g = 2$. The advantage of using Grassmann variables is then seen to be a rather formal one: it allows one in a mathematically rigorous manner to avoid the conclusion that $\boldsymbol{S} = 0$ if $\boldsymbol{S}^2 = 0$. This advantage can be very crucial when studying problems like quantisation (Barducci *et al* 1976) or radiation reaction. Nevertheless, the question whether it is possible to construct an acceptable theory of charged classical point particles with *macroscopic* spin remains open. This might be related to the fact that no consistent (renormalisable) quantum theory of charged particles with arbitrary spin greater than $\hbar$ is known, since the classical theory would be the limit of the quantum theory when $\hbar \to 0$ and $\boldsymbol{S}/\hbar \to \infty$. The failure of all attempts in this direction might indicate a failure of the dogma that structureless particles (or fields) with arbitrary spin exist.

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